Partial differentiation

Maxima Minima for a function of two Variable

$$f(x,y) = 0$$

$f_{xy}) + \dots$

$$\frac{\partial f}{\partial x} = 0 \quad \text{(1)} \qquad \Bigg| \qquad \frac{\partial f}{\partial y} = 0 \quad \text{(2)}$$

Solving (1) and (2) together

$$\left.\begin{matrix} (x_1, y_1) \\ (x_2, y_2) \\ \vdots \end{matrix}\right\} \text{Stationary Points}$$

$$r = \frac{\partial^2 f}{\partial x^2} \qquad s = \frac{\partial^2 f}{\partial x \partial y} \qquad t = \frac{\partial^2 f}{\partial y^2}$$

$\boxed{(x_1, y_1)}$

$$r_{(x_1, y_1)} = \qquad t(x_1, y_1) = \qquad s(x_1, y_1) =$$

Check $(rt - s^2)$

If +Ve then see the signe of (r) & (t)

If the signe of (r) & (t) are +ve then the point will be of minima.

If the sign of r & t is −Ve then it will be the point of maxima

If $rt - s^2 \leq 0$

then we say that the point is saddle point i.e. Neither maxima nor minima.

Repeat It

Q. Locate the stationary points of
$x^4 + y^4 - 2x^2 + 4xy - 2y^2$
and determine their Nature

Solution :-

$$f = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y \qquad \frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$$4x^3 - 4x + 4y = 0 \quad \text{(1)} \qquad 4y^3 + 4x - 4y = 0 \quad \text{(2)}$$

(1) + (2)

$$4x^3 - \cancel{4x} + \cancel{4y} + 4y^3 + \cancel{4x} - \cancel{4y} = 0$$

$$4(x^3 + y^3) = 0$$

$$x^3 + y^3 = 0 \Rightarrow \boxed{x^3 = -y^3}$$

$y = -x$ Satisfies the above reln

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Putting in ①

$4x^3 - 4x - 4x = 0$

$4x^3 - 8x = 0$

$4x(x^2 - 2) = 0$

$x = 0, \quad x^2 - 2 = 0$

$x = 0, \quad x^2 = 2$

$x = 0, \quad x = \pm\sqrt{2}$

$x = 0, \quad x = \sqrt{2}, \quad x = -\sqrt{2}$

$y = 0, \quad y = -\sqrt{2}, \quad y = +\sqrt{2}$

$(0,0) \quad (\sqrt{2}, -\sqrt{2}) \quad (-\sqrt{2}, \sqrt{2})$

Putting in ②

$-4x^3 + 4x - 4(-x) = 0$

$-4x^3 + 4x + 4x = 0$

$-4x^3 + 8x = 0$

$-4x(x^2 - 2) = 0$

$x = 0, \quad x^2 - 2 = 0$

$x = 0, \quad x = \pm\sqrt{2}$

$x = 0, \quad x = \sqrt{2}, \quad x = -\sqrt{2}$

$y = 0, \quad y = -\sqrt{2}, \quad y = +\sqrt{2}$

$(0,0) \quad (\sqrt{2}, -\sqrt{2}) \quad (-\sqrt{2}, \sqrt{2})$

$\therefore$ Three stationary points

$\boxed{(0,0)}, \boxed{(\sqrt{2}, -\sqrt{2})}, \boxed{(-\sqrt{2}, \sqrt{2})}$

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$t = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

$\boxed{(0,0)}$

$r_{(0,0)} = -4$

$s_{(0,0)} = 4$

$t_{(0,0)} = -4$

check

$rt - s^2$

$(-4)(-4) - 4$

$= 16 - 4 = 12$

$(+ve)$

Sign of r & t are $-ve$ $\therefore$ $(0,0)$ is a point of maxima

Maximum value is

$f(0,0) = 0$

$\boxed{(\sqrt{2}, -\sqrt{2})}$

$r_{(\sqrt{2}, -\sqrt{2})} = 12(\sqrt{2})^2 - 4$

$= 24 - 4 = +(20)$

$s_{(\sqrt{2}, -\sqrt{2})} = 4$

$t_{(\sqrt{2}, -\sqrt{2})} = 12(-\sqrt{2})^2 - 4$

$= 24 - 4$

$= (20)$

Check $(rt - s^2)$

$= (20)(20) - (4)^2$

$= 400 - 16 = 384$ $(+ve)$

$\therefore$ see the sign of r & t

$\therefore$ It is a point of minima

$(\sqrt{2}, -\sqrt{2}) \rightarrow$ min

Minimum Value

$f(\sqrt{2}, -\sqrt{2}) = \left((\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(\sqrt{2})(-\sqrt{2}) - 2(-\sqrt{2})^2\right)$

$\boxed{(-\sqrt{2}, \sqrt{2})}$

$r(-\sqrt{2}, \sqrt{2}) = 12(-\sqrt{2})^2 - 4 = 24 - 4 = 20$

$s(-\sqrt{2}, \sqrt{2}) = 4$

$t(-\sqrt{2}, \sqrt{2}) = 12(\sqrt{2})^2 - 4 = 24 - 4 = 20$

Checking

$rt - s^2 = (20)(20) - (4)^2$

$= 400 - 16 = (384) = +ve$

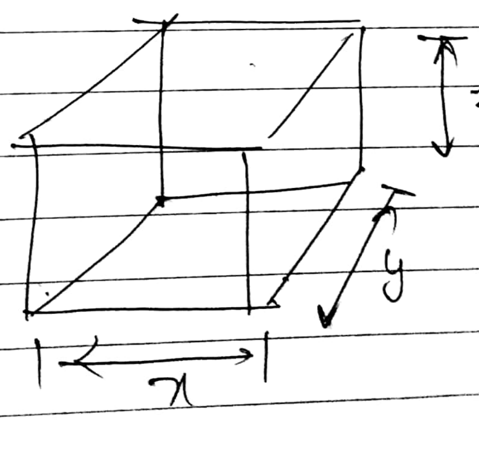
r & t are +ve

$\therefore (-\sqrt{2}, \sqrt{2})$ is a point of minima

Word problem

Q. A rectangular box open at the top is having the given capacity (V), find out the dimensions of the box for least material of construction.

Solution :-



V = given

$V = xyz$

$$\boxed{z = \frac{V}{xy}} \quad \text{--- (1)}$$

Surface area

$$S = xy + 2xz + 2yz$$

$$S = xy + 2z(x+y)$$

using ①

$$S = xy + 2\left(\frac{V}{xy}\right)(x+y)$$

$$S = xy + 2V\left(\frac{1}{y} + \frac{1}{x}\right)$$

Differentiating w.r.t. (x)

$$\frac{\partial S}{\partial x} = y + 2V\left(-\frac{1}{x^2}\right)$$

$$\frac{\partial S}{\partial x} = 0$$

$$y - \frac{2V}{x^2} = 0$$

$$y = \frac{2V}{x^2}$$

$$x^2y = 2V \quad \text{—①}$$

$$\frac{\partial S}{\partial y} = x + 2V\left(-\frac{1}{y^2}\right)$$

$$\frac{\partial S}{\partial y} = 0$$

$$x - \frac{2V}{y^2} = 0$$

$$x = \frac{2V}{y^2}$$

$$xy^2 = 2V \quad \text{—②}$$

Equating ① & ②

$$x^2y = xy^2$$

$$x^2y - xy^2 = 0$$

$$xy(x-y) = 0$$

$\boxed{x=0}$, $\boxed{y=0}$, $x-y=0$
× × $\boxed{y=x}$

Putting in ①

$x^3 = 2V$ $y^3 = 2V$

$x = (2V)^{1/3}$ $y = (2V)^{1/3}$

$r = \frac{\partial^2 S}{\partial x^2} = \frac{4V}{x^3}$

$t = \frac{\partial^2 S}{\partial y^2} = \frac{4V}{y^3} \Rightarrow$

$s = \frac{\partial^2 S}{\partial x \partial y} = 1$

$t = \frac{4V}{((2V)^{1/3})^3} = \frac{4V}{2V} = 2$

$r_{((2V)^{1/3}, (2V)^{1/3})} = 2$

$t_{((2V)^{1/3}, (2V)^{1/3})} = 2$

$s = 1$

$rt - s^2 = (2)(2) - 1 = 3 \; (+ve)$
Check r & $t \Rightarrow (+ve)$

$\therefore$ It is a point of minima.

$\therefore x = (2V)^{1/3}, y = (2V)^{1/3}$

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Lagrange method of undetermined multiplier

Q. Find out the minimum value of $x^2+y^2+z^2$ when $x+y+z=3a$

Solution:-

Let $f = x^2+y^2+z^2$ | $\phi = x+y+z-3a = 0$

Let

$$F = f + \lambda\phi$$

where λ = undetermined multiplier

$$F = (x^2+y^2+z^2) + \lambda(x+y+z-3a)$$

Differentiating w.r.t (x)

$$\frac{\partial F}{\partial x} = (2x) + \lambda(1) \Rightarrow \frac{\partial F}{\partial x} = 0$$

$$2x + \lambda = 0$$

$$\boxed{x = -\lambda/2}$$

Similarly $\frac{\partial F}{\partial y} = 0$

$$\boxed{y = -\lambda/2}$$

$\frac{\partial F}{\partial z} = 0$ & $\boxed{z = -\lambda/2}$

∴ we have

$$x+y+z = 3a$$

$$-\frac{\lambda}{2}-\frac{\lambda}{2}-\frac{\lambda}{2} = 3a$$

$$-\frac{\cancel{3}\lambda}{2} = \cancel{3}a$$

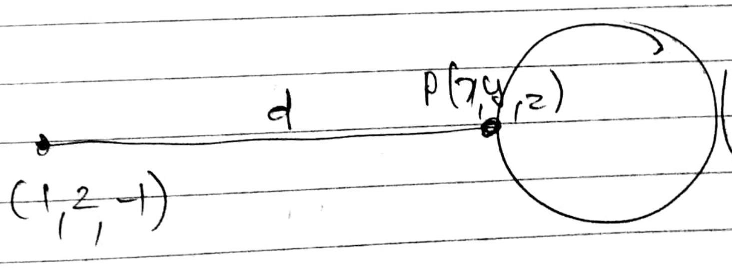
$$\boxed{\lambda = -2a}$$

$$\therefore x = -\frac{(-2a)}{2} = a,\quad y = a,\quad z = a$$

∴ Point is (a, a, a) Ans

Qno Find out the shortest & the longest distance from the point $(1,2,-1)$ to the sphere
$$x^2+y^2+z^2 = 24$$

Solution :-



$\boxed{x^2+y^2+z^2 = 24}$

$$d = \sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$$

Squaring

$$f = d^2$$

$$\boxed{f = (x-1)^2+(y-2)^2+(z+1)^2} \quad \text{—— (1)}$$

$$\boxed{\phi = x^2+y^2+z^2-24} \quad \text{—— (2)}$$

$$F = f + \lambda \phi$$

$$F = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24)$$

$$\frac{\partial F}{\partial x} = 0$$

$$2(x-1) + \lambda(2x) = 0$$

$$2x + \lambda 2x = 2$$

$$x = \frac{2}{2+2\lambda}$$

$$\boxed{x = \frac{1}{1+\lambda}}$$

$$\frac{\partial F}{\partial y} = 0$$

$$2(y-2) + \lambda(2y) = 0$$

$$2y(1+\lambda) = 4$$

$$\boxed{y = \frac{2}{1+\lambda}}$$

$$\frac{\partial F}{\partial z} = 0$$

$$2(z+1) + \lambda 2z = 0$$

$$2z(1+\lambda) = -2$$

$$\boxed{z = \frac{-1}{1+\lambda}}$$

we know that

$$x^2 + y^2 + z^2 = 24$$

$$\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 + \left(\frac{-1}{1+\lambda}\right)^2 = 24$$

$$\frac{1+4+1}{(1+\lambda)^2} = 24$$

$$\frac{\cancel{6}}{(1+\lambda)^2} = \cancel{24}\ 4$$

$$(1+\lambda)^2 = \frac{1}{4}$$

$$1+\lambda = \ldots$$

$$\lambda = \pm\frac{1}{2} - 1$$

$$\lambda = \frac{1}{2} - 1, \quad \lambda = -\frac{1}{2}$$

$$\lambda = -\frac{1}{2}, \quad \lambda = -\frac{3}{2}$$

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Ans-1

$\lambda = -\frac{1}{2}$

$x = 2$

$y = 4$

$z = -2$

$(2, 4, -2)$

$d_1 = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2}$

$= \rightsquigarrow$

minima

Ans-2

$\lambda = -\frac{3}{2}$

$x = -2$

$y = -4$

$z = +2$

$(-2, -4, 2)$

$d_2 = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2}$

$= \rightsquigarrow$

maxima