Maths - II

Partial Differentiation :-

$f = x^2 y^3$

$\frac{\partial f}{\partial x} = 2x y^3$

$\frac{\partial f}{\partial y} = x^2 (3y^2)$

$\frac{\partial^2 f}{\partial x^2} = 2y^3$

$\frac{\partial^2 f}{\partial y^2} = x^2 (6y)$

$$\left\{ \begin{array}{l} \frac{\partial^2 f}{\partial x \partial y} = 2x(3y^2) = 6xy^2 \\ \\ \frac{\partial^2 f}{\partial y \partial x} = (2x)(3y^2) = 6xy^2 \end{array} \right.$$

It shows that in partial differentiation order does not matter.

Que If

$z(x+y) = x^2 + y^2$

Show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

Solution :-

$\therefore$ We have

$$z(x+y) = x^2+y^2$$

$$z = \frac{x^2+y^2}{x+y}$$

$$\frac{\partial z}{\partial x} = \frac{(2x)(x+y)-(1)(x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = \frac{(2y)(x+y)-(1)(x^2+y^2)}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2xy+2y^2-x^2-y^2}{(x+y)^2} \qquad \left(\because \left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}\right)$$

$$\frac{\partial z}{\partial y} = \frac{2xy+y^2-x^2}{(x+y)^2} \quad \text{--- (2)}$$

$$LHS = \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2$$

$$= \left(\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{2xy+y^2-x^2}{(x+y)^2} \right)^2$$

$$= \left(\frac{x^2+\cancel{2xy}-y^2-\cancel{2xy}-y^2+x^2}{(x+y)^2} \right)^2$$

$$= \left(\frac{2x^2-2y^2}{(x+y)^2} \right)^2$$

$$= \left(\frac{2(x-y)\cancel{(x+y)}}{(x+y)^{\cancel{2}}} \right)^2 = \frac{4(x-y)^2}{(x+y)^2} \quad — (3)$$

$$RHS = 4\left(1 - \frac{x^2+2xy-y^2}{(x+y)^2} + \frac{(2xy+y^2-x^2)}{(x+y)^2} \right)$$

$$= 4\left(1 - \left[\frac{\cancel{x^2}+2xy-\cancel{y^2}+2xy+\cancel{y^2}-\cancel{x^2}}{(x+y)^2} \right] \right)$$

$$= 4\left[1 - \frac{4xy}{(x+y)^2} \right]$$

$$= 4\left(\frac{(x+y)^2-4xy}{(x+y)^2} \right) = 4\left[\frac{x^2+y^2+2xy-4xy}{(x+y)^2} \right]$$

$$= 4\left[\frac{x^2+y^2-2xy}{(x+y)^2}\right]$$

$$= \frac{4(x-y)^2}{(x+y)^2}$$

$$= \text{RHS}$$

Proved

Q. Given that

$$u = f(r)$$

where

$$r^2 = x^2+y^2+z^2$$

Show that

$$u_{xx} + u_{yy} + u_{zz} = f''(r) + \frac{2}{r} f'(r)$$

Solution:

We are having

$$u = f(r)$$

Diff w.r.t (x)

$$u_x = f'(r) \times \frac{\partial r}{\partial x}$$

$$\boxed{r^2 = x^2+y^2+z^2}$$

Differentiating w.r.t (x)

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{--- (1)}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{--- (2)}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r} \quad \text{--- (3)}$$

$$u_x = f'(r)\,\frac{x}{r}$$

$$u_x = x\, f'(r)\,\frac{1}{r}$$

Diff. Again w.r.t. (x); Applying product rule :-

$$u_{xx} = (1)\left(\frac{f'(r)}{r}\right) + \left(\frac{x}{r}\right) f''(r)\frac{\partial r}{\partial x} + x\, f'(r)\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right)$$

$$u_{xx} = \frac{f'(r)}{r} + \frac{x}{r} f''(r)\frac{x}{r} - \frac{x}{r^2} f'(r)\frac{x}{r}$$

$$u_{xx} = \frac{f'(r)}{r} + \frac{x}{r} f''(r)\frac{x}{r} - \frac{x^2}{r^3} f'(r) \quad \text{— (4)}$$

Similarly

$$u_{yy} = \frac{f'(r)}{r} + \frac{y}{r} f''(r)\frac{y}{r} - \frac{y^2}{r^3} f'(r) \quad \text{— (5)}$$

Similarly

$$u_{zz} = \frac{f'(r)}{r} + \frac{z}{r} f''(r)\frac{z}{r} - \frac{z^2}{r^3} f'(r) \quad \text{— (6)}$$

$$LHS = u_{xx} + u_{yy} + u_{zz}$$

$$= 3\frac{f'(r)}{r} + \frac{f''(r)}{r^2}(x^2+y^2+z^2) - \frac{f'(r)}{r^3}(x^2+y^2+z^2)$$

$$= 3\frac{f'(r)}{r} + \frac{f''(r)}{\cancel{r^2}}(\cancel{r^2}) - \frac{f'(r)}{r^{\cancel{3}}}(\cancel{r^2})$$

$$= \frac{2 f'(r)}{r} + f''(r)$$

$= RHS$

Proved.

HOMOGENEOUS FUNCTION

$$u = \frac{x^3 + y^3}{\sqrt{x} + \sqrt{y}}$$

$$\frac{x^3}{x^{1/2}} = x^{3-\frac{1}{2}} = x^{5/2}$$

$$\boxed{n = 5/2}$$

$$u = x^2y^2 + xy^3 + x^3y + y^4 \quad n = 4$$

$$u = \frac{x^{1/2} + y^{1/2}}{x^{1/3} \; y^{1/3}}$$

$$\frac{x^{1/2}}{x^{1/3}}$$

$$x^{\frac{1}{2} - \frac{1}{3}}$$

$$x^{1/6}$$

$$\boxed{n = \frac{1}{6}}$$

Q Euler's theorem of Homogeneous function

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Let u is a homogeneous function of degree (n) then

(I) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

(II) $x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=n(n-1)u$

If there is a function of (u) then

i.e $f(u)$

then

(I) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{nf(u)}{f'(u)}$

(II) $x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=g(u)\left(g'(u)-1\right)$

where

$$g(u)=\frac{nf(u)}{f'(u)}$$

Q. If $u=\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Show that

(1) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin(2u)$

(2) $x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=\sin 2u\left(1-4\sin^2 u\right)$

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Sol^n (1)

$$\tan u = \frac{x^3+y^3}{x-y} \qquad \frac{x^3}{x} = x^2$$

$\therefore f(u) = \tan u$

degree

$n = 2$

$\therefore$ By Euler's formula

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}$$

$$= \frac{2(\tan u)}{(\sec^2 u)}$$

$$= 2\frac{\sin u}{\cancel{\cos u}}\cos^{\cancel{2}} u$$

$$= 2\sin u \cos u$$

$$= \sin 2u$$

(2) We know by Euler's formula

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = g(u)\left(g'(u) - 1\right) \quad \text{--- (1)}$$

where

$$g(u) = \frac{n f(u)}{f'(u)}$$

$$g(u) = \frac{2\tan u}{\sec^2 u} = \frac{2\sin u}{\cos u}\cos^2 u$$

$$= 2\sin u\cos u$$

$$= \sin 2u$$

$\therefore$ $$\text{Ans} = g(u)\left(g'(u) - 1\right)$$

$$= (\sin 2u)\left(2\cos 2u - 1\right)$$

$$= 2\sin 2u\cos 2u - \sin 2u$$

$$= \sin(4u) - \sin(2u)$$

$$= 2\cos\left(\frac{4u+2u}{2}\right)\sin\left(\frac{4u-2u}{2}\right)$$

$$= 2\cos\left(\frac{6u}{2}\right)\sin\left(\frac{2u}{2}\right)$$

$$= 2\cos(3u)\sin(u)$$

$$= \text{RHS} \quad \text{proved.}$$

COMPOSITE FUNCTION

Let $f(u,v) = 0$

Differentiating w.r.t. (x)

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\right) = 0$$

Q. If $u = f(x-y, y-z, z-x)$

Show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Solution! -

Here

$$u = f(t_1, t_2, t_3)$$

$$t_1 = x-y, \quad t_2 = y-z, \quad t_3 = z-x$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial f}{\partial t_2}\cdot\frac{\partial t_2}{\partial x} + \frac{\partial f}{\partial t_3}\cdot\frac{\partial t_3}{\partial x}$$

$$= \frac{\partial f}{\partial t_1}(1) + \frac{\partial f}{\partial t_2}(0) + \frac{\partial f}{\partial t_3}(-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial t_1} - \frac{\partial f}{\partial t_3} \quad \text{---} (1)$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial f}{\partial t_2}\cdot\frac{\partial t_2}{\partial y} + \frac{\partial f}{\partial t_3}\cdot\frac{\partial t_3}{\partial y}$$

$$= \frac{\partial f}{\partial t_1}(-1) + \frac{\partial f}{\partial t_2}(1) + 0 = \left(-\frac{\partial f}{\partial t_1} + \frac{\partial f}{\partial t_2}\right)$$

Differentiating w.r.t (z)

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial t_1}\cdot\frac{\partial t_1}{\partial z} + \frac{\partial f}{\partial t_2}\cdot\frac{\partial t_2}{\partial z} + \frac{\partial f}{\partial t_3}\cdot\frac{\partial t_3}{\partial z}$$

$$= \frac{\partial f}{\partial t_1}(0) + \frac{\partial f}{\partial t_2}(-1) + \frac{\partial f}{\partial t_3}(1)$$

$$\frac{\partial u}{\partial z} = -\frac{\partial f}{\partial t_2} + \frac{\partial f}{\partial t_3} \quad \text{—— (3)}$$

~~putting from (1), (2) and (3) in LHS~~

$$LHS = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \cancel{\frac{\partial f}{\partial t_1}} - \cancel{\frac{\partial f}{\partial t_3}} - \cancel{\frac{\partial f}{\partial t_1}} + \cancel{\frac{\partial f}{\partial t_2}} - \cancel{\frac{\partial f}{\partial t_2}} + \cancel{\frac{\partial f}{\partial t_3}}$$

$$= 0 = RHS$$

Q. $u = x^2 - y^2$

$v = 2xy$ $\qquad f(u, v)$

show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$$

Sol^n :-

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\left(\frac{\partial u}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\right)$$

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$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(+2y)$$

$$\frac{\partial f}{\partial x} = 2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v} \quad \text{--- (1)}$$

$$\frac{\partial}{\partial x}(f) = \left(2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right)(f)$$

$$\frac{\partial}{\partial x} = \left(2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right) \quad \text{--- (2)}$$

Now we have to find

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \left(2x\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}\right)\cdot\left(2x\frac{\partial f}{\partial u} + 2y\frac{\partial f}{\partial v}\right)$$

$$= 4x^2\frac{\partial^2 f}{\partial u^2} + 4xy\frac{\partial^2 f}{\partial u\partial v} + 4xy\frac{\partial^2 f}{\partial v\partial u} + 4y^2\frac{\partial^2 f}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial x^2} = 4x^2\frac{\partial^2 f}{\partial u^2} + 8xy\frac{\partial^2 f}{\partial u\partial v} + 4y^2\frac{\partial^2 f}{\partial v^2} \quad \text{--- (3)}$$

Proceeding similarly.

$$\frac{\partial^2 f}{\partial y^2} = 4y^2\frac{\partial^2 f}{\partial u^2} - 8xy\frac{\partial^2 f}{\partial u\partial v} + 4x^2\frac{\partial^2 f}{\partial v^2} \quad \text{--- (4)}$$

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$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (4x^2 + 4y^2)\frac{\partial^2 f}{\partial u^2} + (4y^2 + 4x^2)\frac{\partial^2 f}{\partial v^2}$$

$$= 4(x^2 + y^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) = RHS$$

Proved.