POLAR ASYMPTOTES

(1)

Given the eqn. of the Polar curve as
$$(1/r) = f(\theta)$$

then solve the eqn
$$f(\theta) = 0$$
and find its roots say
$$\theta = \alpha_1, \quad \theta = \alpha_2, \; \text{------}$$

then the eqn of the polar curve will be

(1) $$r \sin(\theta - \alpha_1) = \frac{1}{f'(\alpha_1)}$$

(2) $$r \sin(\theta - \alpha_2) = \frac{1}{f'(\alpha_2)}$$

$$\vdots$$

Q. Find all the asymptotes of the polar curve
$$r(\theta^2 - \pi^2) = 2a\theta$$

Soln :-

$$r = \frac{2a\theta}{\theta^2 - \pi^2}$$

Reciprocal
$$\frac{1}{r} = \frac{\theta^2 - \pi^2}{2a\theta}$$

It is of the form
$$\frac{1}{r} = f(\theta)$$

$$\therefore \; f(\theta) = \frac{\theta^2 - \pi^2}{2a\theta}$$

$\therefore$ Putting
$$f(\theta) = 0$$
$$\frac{\theta^2 - \pi^2}{2a\theta} = 0$$
$$\theta^2 - \pi^2 = 0$$
$$\Rightarrow \theta^2 = \pi^2$$

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$\theta = \pm\pi$

$\theta_1 = \pi$ $\qquad$ $\theta_2 = -\pi$

Polar asymptote-1

$$f(\theta) = \frac{\theta^2 - \pi^2}{2a\theta}$$

$$f'(\theta) = \frac{(2\theta)(2a\theta) - (2a)(\theta^2 - \pi^2)}{(2a\theta)^2}$$

$$f'(\theta) = \frac{4a\theta^2 - 2a\theta^2 + 2a\pi^2}{4a^2\theta^2}$$

$$\therefore f'(\pi) = \frac{4a\pi^2 - \cancel{2a\pi^2} + \cancel{2a\pi^2}}{4a^2\pi^2}$$

$$= \frac{\cancel{4}a\cancel{\pi^2}}{\cancel{4}a^2\cancel{\pi^2}}$$

$$f'(\pi) = \frac{1}{a}$$

Eq^n of polar asymptote is

$$r\sin(\theta - \pi) = \frac{1}{f'(\pi)}$$

$$\boxed{r\sin(\theta - \pi) = a}$$

Polar Asymptote-2

$$f(\theta) = \frac{\theta^2 - \pi^2}{2a\theta}$$

Similarly

$$f'(-\pi) = \frac{4a\pi^2 - \cancel{2a\pi^2} + \cancel{2a\pi^2}}{4a^2\pi^2}$$

$$= \frac{\cancel{4}a\cancel{\pi^2}}{\cancel{4}a^2\cancel{\pi^2}}$$

$$f'(-\pi) = \frac{1}{a}$$

Similarly

$$r\sin(\theta - (-\pi)) = \frac{1}{f'(-\pi)}$$

$$r\sin(\theta + \pi) = \frac{1}{(1/a)}$$

$$\boxed{r\sin(\theta + \pi) = a}$$

Q.2 Find the polar asymptotes of the curve

$$r = \frac{2a}{1-2\cos\theta}$$

Sol^n :- We have

$$r = \frac{2a}{1-2\cos\theta}$$

Reciprocal

$$\frac{1}{r} = \frac{1-2\cos\theta}{2a} = f(\theta)$$

$$\therefore \quad f(\theta) = \frac{1-2\cos\theta}{2a}$$

Putting $f(\theta) = 0$

$$\frac{1-2\cos\theta}{2a} = 0$$

$$1-2\cos\theta = 0$$

$$1 = 2\cos\theta$$

$$\cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\pi/3$$

$$\theta = \left(2n\pi \pm \frac{\pi}{3}\right)$$

$$\boxed{\theta_1 = \frac{\pi}{3}}$$

$$\underbrace{\theta = 2\pi + \frac{\pi}{3}}$$

greater than 2π

$\therefore$ Don't take

$$\theta = 2\pi - \frac{\pi}{3}$$

$$\theta = \frac{6\pi - \pi}{3}$$

$$\boxed{\theta_2 = \frac{5\pi}{3}}$$

You can also find these values of θ between 0 & 2π by calculator also

Ist asymptote is :-

$$r\sin(\theta - \theta_1) = \frac{1}{f'(\theta_1)}$$

$$r \sin\left(\theta - \frac{\pi}{3}\right) = \frac{1}{f'(\pi/3)} \quad \text{——} (1)$$

But $\quad f(\theta) = \dfrac{1 - 2\cos\theta}{2a}$

$$f'(\theta) = \frac{0 - 2(-\sin\theta)}{2a}$$

$$f'(\theta) = \frac{\sin\theta}{a}$$

$$f'(\pi/3) = \frac{\sin\pi/3}{a} = \frac{\sqrt{3}/2}{a}$$

$$f'(\pi/3) = \frac{\sqrt{3}}{2a}$$

$\therefore$ Putting in (1)

$$r \sin\left(\theta - \frac{\pi}{3}\right) = \frac{1}{(\sqrt{3}/2a)}$$

$$\boxed{r \sin\left(\theta - \frac{\pi}{3}\right) = \frac{2a}{\sqrt{3}}} \quad \text{Ans - 1}$$

IInd Polar asymptote :-

$$r \sin(\theta - 5\pi/3) = \frac{1}{f'(5\pi/3)}$$

But $\quad f'(\theta) = \dfrac{\sin\theta}{a}$

$$\therefore f'(5\pi/3) = \frac{\sin(5\pi/3)}{a} = \frac{-\sqrt{3}/2}{a} = \frac{-\sqrt{3}}{2a}$$

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$$\Rightarrow \sin\left(\theta - \frac{5\pi}{3}\right) = \frac{1}{(-\sqrt{3}/2a)}$$

$$\boxed{\Rightarrow \sin\left(\theta - \frac{5\pi}{3}\right) = -\frac{2a}{\sqrt{3}}}$$

Ans - 2

RADIUS OF CURVATURE BY NEWTON'S METHOD

Radius of curvature of a curve passing through origin can be found by Newton's method.

Case-1 when x-axis is the tangent at origin to the curve then

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x^2}{2y} \right)$$

Case-2 when y-axis the tangent to the curve at origin then

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{y^2}{2x} \right)$$

Q. By Newton method find R.O.C. of the following curve at origin

$$y^3 = x^2 + 2xy + y^2 + x$$

Sol^n:-

$$x^2 + 2xy + y^2 + x - y^3 = 0$$

There is no constant term ∴ Curve passes through origin

For tangent at origin put lowest degree term $= 0$

∴ $x = 0$ This is the eq^n of y-axis

Dividing both sides by $(2x)$

$$\frac{x^2}{2x} + \frac{2xy}{2x} + \frac{y^2}{2x} + \frac{x}{2x} - \frac{y^3}{2x} = 0$$

$$\frac{x}{2} + y + (\rho) + \frac{1}{2} - \frac{y}{\cancel{2}}\rho = 0$$

Now Putting $\lim_{\substack{x \to 0 \\ y \to 0}}$ ②

$$0 + 0 + \rho + \frac{1}{2} - 0 = 0$$

$$\boxed{\rho = -\frac{1}{2}}$$ Ans

ALTERNATING SERIES

A series is said to be alternating if its alternating terms are of opposite sign

$$u_1 - u_2 + u_3 - \cdots$$

Leibnitz test of the Alternating series

Condition (1) :- Successive terms must be of decreasing magnitude

$$u_1 > u_2 > u_3 > \cdots$$

Condition (2) :- $\lim_{n\to\infty} (u_n) = 0$

If above two conditions are satisfied then alternating series will be convergent otherwise oscillatory

Qe

$$\frac{1}{1!} - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots \infty$$

Test the nature of above series.

Sol^n :- Condition (1) :-

$$\frac{1}{1!} > \frac{1}{2!} > \frac{1}{4!} > \cdots$$

i.e. successive terms are of decreasing magnitude, hence first condition is satisfied.

Condition (2) :- nth term is

$$u_n = \frac{1}{(2n)!}$$

$$\lim_{n\to\infty} \frac{1}{(2n)!} = \frac{1}{\infty} = 0$$

Both Conditions are satisfied
therefore series is converging by Leibnitz test.

Q.2 Test the nature of following Series

$$\frac{5}{2} - \frac{7}{4} + \frac{9}{6} - \frac{11}{8} + \cdots \infty$$

Soln :- Condition (1) :-

$$\frac{5}{2} > \frac{7}{4} > \frac{9}{6} \cdots\cdots$$

∴ 1st Condition is Satisfied

Condition (2) :-

nth term is

Numerator: 5, 7, 9, 11 ---- (A.P.)

$$a_n = 5 + (n-1)2 = 2n+3$$

$$u_n = \frac{2n+3}{2n}$$

Denominator: 2, 4, 6, ----- (A.P.)

$$a_n = 2 + (n-1)2 = 2n$$

$$\therefore \lim_{n\to\infty} (u_n)$$

$$= \lim_{n\to\infty}\left(\frac{2n+3}{2n}\right) = \lim_{n\to\infty}\left(1 + \frac{3}{2n}\right) = 1 + 0 = 1 \neq 0$$

∴ 2nd Condition of Leibnitz test is not satisfied ∴ Series is oscillatory.

ABSOLUTE CONVERGENCE

Step-1 :- Check the convergence of series by Leibnitz test. If its is convergent go to step-2

Step-2 :- Considering all terms positive, apply conventional test of a positive series to it and then check its convergence

(1) If series is convergent in both steps then we say that the series is absolutely convergent.

(2) If series is convergent in step-1 but divergent in step-2 then it is said to be conditionally convergent

(3) If divergent in step-1 then series will be oscillatory.

Q Check the absolute convergence of the followi[ng] series

$$1 - \frac{1}{2^3}(1+2) + \frac{1}{3^3}(1+2+3) - \frac{1}{4^3}(1+2+3+4) + \cdots \infty$$

Soln :- nth term is

$$u_n = (-1)^{n-1}\frac{1}{n^3}(1+2+3+\cdots+n)$$

$$u_n = (-1)^{n-1}\frac{1}{n^3}\frac{(n(n+1))}{2}$$

$$u_n = \frac{(-1)^{n-1}(n+1)}{2n^2}$$

Step-1 Applying Leibnitz test (4)

(i) $1 > \frac{3}{8} > \frac{6}{27} > \cdots$

Successive terms are decreasing

(ii) $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} \left(\frac{(n+1)}{2n^2}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{2n} + \frac{1}{2n^2}\right)$$

$$= \frac{1}{\infty} + \frac{1}{\infty} = 0 + 0 = 0$$

$\therefore$ Converging by Leibnitz test.

Step-2 :- Considering it as a positive Series

$\therefore$ n^{th} term will be

$$u_n = \frac{(n+1)}{2n^2} \quad \text{——— (1)}$$

By Comparison test, First we Let

$$V_n = \frac{n}{n^2}$$

(Highest degree of Nr. of u_n)

(Highest degree of Dr. of u_n)

$V_n = \frac{1}{n}$ ——— (2) where ΣV_n is diverging by p-series

Dividing (1) by (2)

$$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)$$

$$\lim_{n\to\infty}\left(\frac{\frac{(n+1)}{2n^2}}{\frac{1}{n}}\right)$$

$$\lim_{n\to\infty}\left(\frac{n(n+1)}{2n^2}\right)$$

$$\lim_{n\to\infty}\left(\frac{(n+1)}{2n}\right)$$

$$\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right)$$

$$\frac{1}{2}+0=\frac{1}{2}$$

Which is a finite number

∴ ~~Converging~~ Diverging by Comparison tests.

∴ In step-1 series Converging

In step-2 series Diverging

∴ We say that the series is Conditionally Converging.